

Minimum Volume Ellipsoids

Adam Rahman

November 25, 2024

The problem of finding the ellipsoid of minimum volume containing a set of points $\mathbf{v}_1, \dots, \mathbf{v}_n$ is stated as the following optimization problem ([1])

$$\begin{aligned} & \underset{\mathbf{B}, \mathbf{d}}{\text{maximize}} && \log \det(\mathbf{B}) \\ & \text{subject to} && \|\mathbf{B}\mathbf{x} + \mathbf{d}\| \leq 1, \quad \forall \mathbf{v}_i \in \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \end{aligned}$$

The function `minelips` takes as input an $n \times p$ matrix \mathbf{V} containing the points around which we would like to find the minimum volume ellipsoid, and returns the optimal solution using `sqp`.

```
R> out <- minelips(V)
```

Numerical Example

We consider a small point configuration of size 25 in two dimensions.

```
R> data(Vminelips)
```

```
      V1      V2
[1,]  1.371 -0.430
[2,] -0.565 -0.257
[3,]  0.363 -1.763
[4,]  0.633  0.460
[5,]  0.404 -0.640
[6,] -0.106  0.455
[7,]  1.512  0.705
[8,] -0.095  1.035
[9,]  2.018 -0.609
[10,] -0.063  0.505
[11,]  1.305 -1.717
[12,]  2.287 -0.784
[13,] -1.389 -0.851
[14,] -0.279 -2.414
[15,] -0.133  0.036
[16,]  0.636  0.206
[17,] -0.284 -0.361
[18,] -2.656  0.758
[19,] -2.440 -0.727
[20,]  1.320 -1.368
```

```
[21,] -0.307  0.433
[22,] -1.781 -0.811
[23,] -0.172  1.444
[24,]  1.215 -0.431
[25,]  1.895  0.656
```

```
R> out <- minelips(Vminelips)
```

Here, the output we are interested in \mathbf{B} and \mathbf{d} are stored in the output vector \mathbf{y} , but not in a straightforward way.

```
R> y <- out$y
```

```
      [,1]
[1,] 0.37878339
[2,] 0.48368646
[3,] 0.01425185
[4,] 0.12947058
[5,] 0.17165170
```

```
R> p <- ncol(Vminelips)
```

```
R> B <- diag(y[1:p])
R> tmp <- p
R> for(k in 1:(p-1)){
R>   B[(k+1):p,k] <- y[tmp + c(1:(p-k))]
R>   B[1,(k+1):p] <- B[(k+1):p,k]
R>   tmp <- tmp + p - k
R> }
```

```
R> d <- y[(tmp+1):length(y)]
```

References

- [1] Lieven Vandenberghe, Stephen Boyd, and Shao-Po Wu. Determinant maximization with linear matrix inequality constraints. *SIAM journal on matrix analysis and applications*, 19(2):499–533, 1998.